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q-special functions with $|q| = 1$ and their application to discrete integrable systems

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Abstract
Solutions for certain discrete integrable systems are constructed by using integral solutions for hypergeometric q-difference systems with $|q| = 1$. We apply a solution for Lauricella's D-type hypergeometric q-difference system with $|q| = 1$ to construct a solution for the discrete KP-hierarchy. Furthermore, an integral solution of a q-difference analogue of Bessel's equation is newly introduced and applied to construct a solution for a q-difference analogue of the cylindrical Toda equation with $|q| = 1$.

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1. Introduction

In recent progress on studies on the quantum integrable systems and the representation theory of quantum groups, q-analyses with $|q| = 1$ have become of general interest. They are necessary for massive field theory [27], for XXZ-models in the gapless regime and representation theory on $U_q(sl(2, \boldsymbol{R}))$ and on $SL_q(2; \boldsymbol{R})$. Basic ideas for the construction of integral solutions were introduced by Jimbo–Miwa and by Ruijsenaars. Jimbo and Miwa [5] used Kurokawa's double sine function for the construction of integral solutions for the quantized Kniznik–Zamolodchikov equation with $|q| = 1$ (see also [10, 11, 14, 15]). Ruijsenaars [25] introduced generalized gamma functions and applied them to construct eigenfunctions of commuting difference operators. Jimbo and Miwa's idea is effective for hypergeometric q-difference systems. Nishizawa and Ueno [17–19] constructed the solutions as the Barnes type and the Euler type integrals. Takeyama [28] studied structures of the Barnes type solution from a viewpoint of twisted q-de Rham cohomologies. Ponsot and Teschner [24] and Kharchev *et al* [9] applied Nishizawa and Ueno's solution to representations of $U_q(sl(2, \boldsymbol{R}))$.

We introduce some integral solutions for q-difference systems with $|q| = 1$ and apply them to construct bilinear relations of some discrete integrable systems. In studies on nonlinear integrable systems, various researchers applied special function theory to construct their solutions. Kametaka [7,8], Okamoto [21,22] and Nakamura [16] investigated hypergeometric solutions of the Toda equation. With respect to discrete integrable systems, Kajiwara and Satsuma [6] introduced a q-difference analogue of the cylindrical Toda equation, whose solution is represented by using q-Bessel functions. Tokihiro *et al* [29,31] considered various special function solutions based on the Darboux transformations. They gave a solution for the discrete KP-hierarchy introduced by Miwa [13] and Ohta *et al* [23]. Their method is applicable to the construction in the case when $|q| = 1$. In this paper, we propose to give q-special function solutions for nonlinear discrete integrable systems in the case when $|q| = 1$. In section 2, we give a brief survey on Kurokawa's double sine function and certain functions which play important roles in the following section. In section 3, we construct a τ-function of the discrete KP-hierarchy by using an integral solution for Lauricella's D-type hypergeometric q-difference system with $|q| = 1$ [17]. It satisfies the same contiguous relations as those of q-Lauricella's hypergeometric function with $0 < q < 1$ (cf [20]). We can use the same machinery as Tokihiro *et al* [29] and can construct a special case of the Casorati type solution for the bilinear relation introduced by Miwa [13] and Ohta *et al* [23]. In section 4, we newly introduce an integral solution for a q-difference analogue Bessel's equation with $|q| = 1$ (we call it the 'q-Bessel function with $|q| = 1$'). In the case when $0 < q < 1$, some kinds of q-analogues of the Bessel function are known (see e.g. [2]). Our solution corresponds to a Barnes type integral of Jackson's q-Bessel function [3,4]. We regard contiguous relations of the integral as dispersion relations in a similar way to Kajiwara–Satsuma [6] and construct a solution for their cylindrical q-Toda equation in the case when $|q| = 1$.

2. q-gamma function with $|q| = 1$

In this section, we introduce the function which is important in the following argument. First, we introduce Kurokawa's double sine function $S_2(z|(\omega_1, \omega_2))$ [1,12].

Definition 2.1. *For* $\underline{\omega} := (\omega_1, \omega_2) \in \boldsymbol{C}^2$*, we define* $\zeta_2(s, z|\underline{\omega})$*,* $\Gamma_2(z|\underline{\omega})$ *and* $S_2(z|\underline{\omega})$ *as*

$$\zeta_2(s, z|\underline{\omega}) := \sum_{m_1, m_2 \in \boldsymbol{Z}_{\geqslant 0}} (z + m_1\omega_1 + m_2\omega_2)^{-s}$$

$$\Gamma_2(z|\underline{\omega}) := \exp\left(\frac{\partial}{\partial s}\zeta_2(s, z|\underline{\omega})|_{s=0}\right)$$

$$S_2(z|\underline{\omega}) := \Gamma_2(z|\underline{\omega})^{-1}\Gamma_2(\omega_1 + \omega_2 - z|\underline{\omega}).$$

It is known that the double sine function satisfies the functional relation

$$\frac{S_2(z + \omega_1|\underline{\omega})}{S_2(z|\underline{\omega})} = \frac{1}{2\sin\frac{\pi z}{\omega_2}}. \tag{1}$$

By using this function, we can construct a 'q-shifted factorial with $|q| = 1$'. We suppose that $q = \mathrm{e}^{2\pi \mathrm{i}\omega}$ $(0 < \omega < 1,\ \omega \notin \boldsymbol{Q})$, i.e. $|q| = 1$ and q is not a root of unity. From now on, we take a branch of a logarithm such that $\log q = 2\pi\mathrm{i}\omega$.

Definition 2.2. *We define* $\langle z\rangle$*,* $\tilde{\Gamma}(z)$ *and* $\tilde{B}(z)$ *as*

$$\langle z\rangle = \langle z; q\rangle := i^{z-1}q^{-\frac{z(z-1)}{4}}S_2\left(z\,\middle|\left(1, \frac{1}{\omega}\right)\right)$$

$$\tilde{\Gamma}(z) = \tilde{\Gamma}(z; q) := \sqrt{\omega}^{-1}(1-q)^{1-z}\langle z\rangle^{-1}$$

$$\tilde{B}(a, b) = \tilde{B}(a, b; q) := \frac{\tilde{\Gamma}(a)\tilde{\Gamma}(b)}{\tilde{\Gamma}(a+b)}.$$

These functions have the following properties:

Lemma 2.3. (1) $\langle z\rangle$ *and* $\tilde{\Gamma}(z)$ *satisfy functional equations*

$$\langle z\rangle = (1-q^z)\langle z+1\rangle$$
$$\tilde{\Gamma}(z+1) = [z]_q\tilde{\Gamma}(z) \qquad \tilde{\Gamma}(1) = 1$$

where

$$[x]_q = \frac{1-q^x}{1-q}. \tag{2}$$

(2) $\langle z\rangle$ *has simple poles at*

$$z = n_1 + \frac{n_2}{\omega} \qquad (n_1, n_2 \in \boldsymbol{Z}_{>0})$$

and simple zeros at

$$z = n_1 + \frac{n_2}{\omega} \qquad (n_1, n_2 \in \boldsymbol{Z}_{\leqslant 0}).$$

(3) As $|z| \to \infty$ *within a sector not containing the real axis,* $\langle z\rangle$ *and* $\tilde{\Gamma}(z;q)$ *have asymptotic behaviour:*

$$\langle z\rangle = \begin{cases} \mathrm{O}(1) & \mathrm{Im}\, z > 0 \\ \exp[-\pi\mathrm{i}\{\omega z^2 - (1+\omega)z)\} + \mathrm{O}(1)] & \mathrm{Im}\, z < 0 \end{cases}$$

$$\tilde{\Gamma}(z;q) = \exp\left[(1-z)\log(q-1) + (z-1)\log\mathrm{i} \right.$$
$$\left. + \frac{z(z-1)}{4}\log q \mp \pi\mathrm{i}\left(\frac{\omega z^2}{2} - \frac{\omega+1}{2}z\right) + \mathrm{O}(1)\right] \qquad (\text{for } \pm\,\mathrm{Im}\, z > 0).$$

This lemma follows from results of [5, 26]. We note that $\langle z\rangle$ (resp. $\tilde{\Gamma}(z;q)$) satisfies the same relation as the q-shifted factorial $(q^z)_\infty := \prod_{k=0}^{\infty}(1-q^{z+k})$ (resp. the q-gamma function $\Gamma_q(z) := (1-q)^{1-z}\frac{(q)_\infty}{(q^z)_\infty}$) with $0 < q < 1$. In the case when $|q| = 1$, these infinite products do not converge, however, we can construct an integral solution by using $\langle z\rangle$ and $\tilde{\Gamma}(z;q)$ instead of $(q^z)_\infty$ and $\Gamma_q(z)$.

3. A solution of Lauricella's hypergeometric q-difference system with $|q| = 1$

3.1. Construction of an integral solution

In this section, we introduce a q-special function solution for the discrete KP-hierarchy by using an integral solution for Lauricella's D-type hypergeometric q-difference system with $|q| = 1$ [18]. First we recall the idea how the solution can be constructed. In the case when $0 < q < 1$, the Jackson integral solution for Lauricella's D-type hypergeometric q-difference system [20] is represented as follows:

$$\phi_D(z) = \phi_D\left(\begin{matrix} a; & b_1 & b_2 & \ldots & b_n \\ & & c & & \end{matrix}\; ; \quad z; \quad q\right)$$
$$= \frac{1}{B_q(a, c-a)}\int_0^1 t^a \frac{(tq)_\infty}{(tq^{c-a})_\infty}\prod_{k=1}^{n}\frac{(tz_kq^{b_k})_\infty}{(tz_k)_\infty}\frac{d_qt}{t} \tag{3}$$

where a, b_j $(j = 1, 2, \ldots, n)$ and c are complex parameters, $B_q(x, y)$ is the q-beta function (see [2]). This satisfies the following q-difference system:

$$\{(1 - cq^{-1}T_q)(1 - T_{q,z_j}) - z_j(1 - aT_q)(1 - b_jT_{q,z_j})\}\phi_D(z) = 0 \qquad (j = 1, 2, \ldots n)$$
$$\{z_j(1 - b_jT_{q,z_j})(1 - T_{q,z_k}) - z_k(1 - T_{q,z_j})(1 - b_kT_{q,z_k})\}\phi_D(z) = 0, \qquad (1 \leqslant j < k \leqslant n) \tag{4}$$

where T_{q,z_k} is a q-shift operator acting on z_k and $T_q := T_{q,z_1}T_{q,z_2}\cdots T_{q,z_n}$.

We introduce an integral solution of a q-difference analogue of Lauricella's D-type hypergeometric system with $|q| = 1$ [17]. In this section, we suppose that $q = \mathrm{e}^{2\pi\mathrm{i}\omega}$ $(0 < \omega < 1,\ \omega \notin \boldsymbol{Q})$ and take such a branch of a logarithm that $\log q = 2\pi\mathrm{i}\omega$. In order that the integral makes sense, we impose the following conditions on the parameters a, b_j $(j = 1, 2, \ldots n)$ and $c \in \boldsymbol{C}$:

Conditions on parameters. *We assume that*

$$a - c \notin \boldsymbol{Z}_{>0} \tag{5}$$
$$b_j \notin \boldsymbol{Z}_{<0} \qquad \text{for} \quad j = 1, 2, \ldots, n \tag{6}$$
$$\mathbb{R}a > 0 \qquad \mathbb{R}\left(c - \sum_{j=1}^{n} b_j - 2\right) > 0. \tag{7}$$

Under these conditions, we can define the Euler integral $\Psi_D(x)$. Let us denote by K any bounded domain in the region

$$\{x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{C}^n \mid x_j \notin \boldsymbol{Z}_{<0} \text{ for } j = 1, \ldots, n\}.$$

Definition 3.1. *Suppose that a, b_j and c satisfy (5)–(7) then, for $x \in K$, we define $\Psi_D(x)$ by*

$$\Psi_D(x) = \Psi_D\left(\begin{array}{c} a;\ b_1\ b_2\ \ldots\ b_n \\ c \end{array};\ x;\ q\right)$$
$$:= \frac{1}{\tilde{B}(a, c-a)} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} q^{as} \frac{\langle s+1\rangle}{\langle s+c-a\rangle} \prod_{k=1}^{n} \frac{\langle s+x_k+b_k\rangle}{\langle s+x_k\rangle}\,\mathrm{d}s$$

where the contour lies on the right of the poles

$$s = -x_k + n_1 + \frac{n_2}{\omega} \qquad s = a - c + n_1 + \frac{n_2}{\omega} \qquad (n_1, n_2 \in \boldsymbol{Z}_{\leqslant 0})$$

and on the left of the poles

$$s = -x_k - b_k + m_1 + \frac{m_2}{\omega} \qquad s = -1 + m_1 + \frac{m_2}{\omega} \qquad (m_1, m_2 \in \boldsymbol{Z}_{>0})$$

where $k = 1, 2, \ldots, n$.

Then, we can see that $\Psi_D(x)$ is a solution of the system of difference equations which are obtained by transforming the multiplicative variables of (4) to the additive variables.

Theorem 3.2.

$$\{(1 - q^{c-1}T^+)(1 - T^+_{x_j}) - q^{x_j}(1 - q^aT^+)(1 - q^{b_j}T^+_{x_j})\}\Psi_D(x) = 0 \qquad (j = 1, 2, \ldots n)$$
$$\{q^{x_j}(1 - q^{b_j}T^+_{x_j})(1 - T^+_{x_k}) - q^{x_k}(1 - T^+_{x_j})(1 - q^{b_k}T^+_{x_k})\}\Psi_D(x) = 0 \qquad (1 \leqslant j < k \leqslant n) \tag{8}$$

where

$$(T^+_{x_j}f)(x_1, x_2, \ldots, x_n) := f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n)$$
$$T^+ := T^+_{x_1}T^+_{x_2}\cdots T^+_{x_n} \qquad T^+_{>k} := T^+_{x_{k+1}}\cdots T^+_{x_n}.$$

3.2. A solution for the discrete KP-hierarchy

We note that $\Psi_D(x)$ satisfy similar contiguous relations [18] to those of the q-analogue of Lauricella's D-type hypergeometric function (cf [20]). Thus, we can use the same method as Tokihiro *et al* [29]. If we define $\phi(k_1, k_2, \ldots, k_N; t)$ as

$$\begin{aligned}\phi(k_1, k_2, \ldots, k_N; t) &:= \tilde{B}(a, c-a) \\ &\quad\times \Psi_D\left(\begin{matrix} a+t; & b_1-k_1 & b_2-k_2 & \ldots & b_N-k_N \\ & & c+t & & \end{matrix}\; ; \quad x; \quad q\right) \\ &= \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} q^{(a+t)s} \frac{\langle s+1\rangle}{\langle s+c-a\rangle} \prod_{j=1}^{N} \frac{\langle s+x_j+b_j-k_j\rangle}{\langle s+x_j\rangle}\, \mathrm{d}s\end{aligned}$$

then, we can see that

$$\begin{aligned}&\phi(k_1, k_2, \ldots, k_N; t) - \phi(k_1, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_N; t) \\ &\qquad = q^{x_j+b_j-k_j}\phi(k_1, k_2, \ldots, k_N; t+1).\end{aligned} \tag{9}$$

From now on, we introduce N-sets of parameters $S_i := \{a^{(i)}, \{b_j^{(i)}\}_{2\leqslant i\leqslant N}, c^{(i)}\}$ $(1 \leqslant j \leqslant N)$ such that all S_i satisfy conditions (5)–(7) and that vectors

$$(\phi_i(k_1, k_2, \ldots, k_N; 0), \ldots, \phi_i(k_1, k_2, \ldots, k_N; N-1)) \qquad (1 \leqslant i \leqslant N)$$

are a linearly independent set where

$$\begin{aligned}\phi_i(k_1, k_2, \ldots, k_N; t) &:= \tilde{B}(a^{(i)}, c^{(i)} - a^{(i)}) \\ &\quad\times \Psi_D\left(\begin{matrix} a^{(i)}+t; & b_2^{(i)}-k_2 & b_3^{(i)}-k_3 & \ldots & b_N^{(i)}-k_N \\ & & c^{(i)}+t & & \end{matrix}\; ; \quad x; \quad q\right).\end{aligned}$$

From relation (9), it follows that

$$\det\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{N-2} & a_1^{N-2}\tau_1\hat{\tau}_1 \\ 1 & a_2 & a_2^2 & \cdots & a_2^{N-2} & a_2^{N-2}\tau_2\hat{\tau}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & a_N & a_N^2 & \cdots & a_N^{N-2} & a_N^{N-2}\tau_N\hat{\tau}_N \end{bmatrix} = 0 \tag{10}$$

for $k_j \in \boldsymbol{Z}_{\geqslant 0}$, where

$$\begin{aligned}a_j &:= q^{x_j+b_j-k_j-1} \qquad \text{for} \quad 1 \leqslant j \leqslant N. \\ \tau_i &:= \tau(k_1, k_2, \ldots, k_{i-1}, k_i+1, k_{i+1}, \ldots, k_N) \\ \hat{\tau}_i &:= \tau(k_1+1, k_2+1, \ldots, k_{i-1}+1, k_i, k_{i+1}+1, \ldots, k_N+1) \\ \tau(k_1, k_2, \ldots, k_N) &:= \det\,[\phi_i(k_1, k_2, \ldots, k_N; j)]_{0\leqslant i.j\leqslant N}\,.\end{aligned}$$

Relation (10) is a special case of an autonomous version of the bilinear relation of the discrete KP-hierarchy introduced in Miwa [13] and Ohta *et al* [23]. We obtain a q-special function solution in the case when $|q| = 1$.

4. q-Bessel function with $|q| = 1$

4.1. Construction of an integral solution

In this section, we construct an integral solution for a q-difference analogue of Bessel's equation in the case when $|q| = 1$. First, let us recall the case when $0 < q < 1$. We introduce a q-difference analogue of Bessel's equation as

$$\left([\vartheta+\alpha]\,[\vartheta-\alpha] + z^2\right) f(z) = 0 \tag{11}$$

where

$$[\vartheta + a]\, f(z) := \frac{1 - q^a f(qz)}{1-q}.$$

As $q \to 1$, $[\vartheta + \alpha]$ coincides with the Euler operator

$$(\vartheta + \alpha) = z\frac{\mathrm{d}}{\mathrm{d}z} + \alpha$$

and (11) coincides with Bessel's differential equation (cf [30])

$$\left\{ z^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2} + z\frac{\mathrm{d}}{\mathrm{d}z} + \left(z^2 - \alpha^2\right) \right\} f(z) = 0.$$

A solution of (11) can be represented as the following power series:

$$f(z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^{\alpha+2k}}{[2]_q^{\alpha+2k}\Gamma_{q^2}(\alpha+k+1)\Gamma_{q^2}(k+1)}. \tag{12}$$

By using residue calculus, we can see that the above series $f(z)$ has an integral representation

$$f(z) = \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{1}{[2]_q^{\alpha+2s}\Gamma_{q^2}(\alpha+s+1)\Gamma_{q^2}(s+1)} \frac{\pi z^{\alpha+2s}}{\sin \pi s}\, \mathrm{d}s. \tag{13}$$

In the case when $|q| = 1$, the power series (12) does not converge. However, we can obtain an integral solution like (13). We construct the integral by using $\tilde{\Gamma}(z; q)$ instead of $\Gamma_q(z)$.

$$\int \frac{1}{[2]_q^{\alpha+2s}\tilde{\Gamma}(\alpha+s+1; q^2)\tilde{\Gamma}(s+1; q^2)} \frac{\pi z^{\alpha+2s}}{\sin \pi s}\, \mathrm{d}s.$$

We note that the integral satisfies (11) if $\tilde{\Gamma}(z; q)$ satisfies lemma 2.3 (2) and if the integral converges. Therefore, under the suitable condition on the parameter α, we can obtain an integral solution for (11) even in the case $|q| = 1$.

From now on, we suppose $q := \mathrm{e}^{\pi \mathrm{i}\omega}$ $(0 < \omega < 1,\ \omega \notin \boldsymbol{Q})$ and that log takes such a branch that $\log q = \pi \mathrm{i}\omega$. We introduce an integral solution of a q-difference analogue of Bessel's equation. In order that the integral makes sense, we impose a condition on the parameter α:

$$\operatorname{Re}\alpha > 1. \tag{14}$$

Under this condition, let us define a real number δ such that

$$0 < \delta < \frac{\pi\omega}{2}(\operatorname{Re}\alpha - 1)$$

and a sector

$$S_\delta := \left\{ z \in \boldsymbol{C} \,\middle|\, -\frac{\pi\omega}{2} + \delta < \arg(z) < \pi\omega \operatorname{Re}\alpha - \frac{3\pi\omega}{2} - \delta \right\}.$$

We introduce an integrand function $j_\alpha(s, z; q)$ and an integral $J_\alpha(z; q)$ as follows:

Definition 4.1.

$$j_\alpha(s, z; q) := \frac{1}{[2]_q^{\alpha+2s}\tilde{\Gamma}(\alpha+s+1; q^2)\tilde{\Gamma}(s+1; q^2)} \frac{\pi z^{\alpha+2s}}{\sin \pi s}$$

$$J_\alpha(z; q) := \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} j_\alpha(s, z; q)\, \mathrm{d}s$$

where the contour lies on the left of the poles

$$s = m_1 + \frac{m_2}{\omega} \qquad \left(m_1 \in \boldsymbol{Z}_{\geqslant 0}, m_2 \in \boldsymbol{Z}_{\geqslant 0}\right)$$

$$s = -\alpha + m_1 + \frac{m_2}{\omega} \qquad \left(m_1 \in \boldsymbol{Z}_{\geqslant 0}, m_2 \in \boldsymbol{Z}_{>0}\right).$$

From lemma 2.3, $j_a(s, z; q)$ decays exponentially as $s \to \pm i\infty$. Thus, $J_\alpha(z; q)$ is analytic in the sector S_δ and can be continued analytically. We can see that $J_\alpha(z; q)$ satisfies a q-difference analogue of Bessel's equation with $|q| = 1$.

Theorem 4.2.

$$\{[\vartheta + \alpha][\vartheta - \alpha] + z^2\} J_\alpha(z; q) = 0.$$

Proof. This theorem can be proved in a similar way to Nishizawa–Ueno [18]. We note that $j_\alpha(s, z; q)$ satisfies

$$[\vartheta + \alpha]\, [\vartheta - \alpha]\, j_\alpha(s, z; q) = -z^2 j_\alpha(s + 1, z; q).$$

Thus, from the location of the poles of the integrand, we have

$$[\vartheta + \alpha]\, [\vartheta - \alpha] \int_{-i\infty}^{+i\infty} j_\alpha(s, z; q)\, ds = -z^2 \int_{-i\infty}^{+i\infty} j_\alpha(s, z; q)\, ds$$

because these q-difference operators commute the integral. □

4.2. *Solution for the q-Toda equation with* $|q| = 1$

We can see that $J_\alpha(z; q)$ satisfies the contiguous relations

$$[\vartheta - \alpha]\, J_\alpha(z; q) = -z J_{\alpha+1}(z; q) \qquad [\vartheta + \alpha]\, J_\alpha(z; q) = z J_{\alpha-1}(z; q) \qquad (15)$$

by using the same argument as the proof of theorem 4.2. Once we have these relation, we can apply Kajiwara–Satsuma's method [6] to construct a solution for a q-difference analogue of the cylindrical Toda equation with $|q| = 1$.

We define $\tau_n(r)$ as

$$\tau_n(r) := \det[J_{n+p_i+j-1}(r; q)]_{1\leqslant i,j\leqslant N}$$

where p_i $(i = 1, \ldots, N)$ are such constants that

$$\mathrm{Re}\, p_i > 1$$

and that the above determinant does not vanish. For $n \geqslant 1$, $\tau_n(r)$ satisfies the following bilinear equation:

$$\tau_n(q^2 r)\tau_n(r) - \tau_n^2(qr) = (1 - q)^2 r^2 \{\tau_{n+1}(qr)\tau_{n-1}(qr) - \tau_n(q^2 r)\tau_n(r)\}.$$

This is a bilinear relation for a q-difference analogue of the cylindrical Toda equation. We have seen that a Kajiwara–Satsuma type solution can be constructed even in the case when $|q| = 1$.

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